Mutually catalyzed birth of population and assets in exchange-driven growth

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We propose an exchange-driven aggregation growth model of population and assets with mutually catalyzed birth to study the interaction between the population and assets in their exchange-driven processes. In this model, monomer (or equivalently, individual) exchange occurs between any pair of aggregates of the same species (population or assets). The rate kernels of the exchanges of population and assets are $K(k,l)=Kkl$ and $L(k,l)=Lkl$, respectively, at which one monomer migrates from an aggregate of size k to another of size l. Meanwhile, an aggregate of one species can yield a new monomer by the catalysis of an arbitrary aggregate of the other species. The rate kernel of asset-catalyzed population birth is $I(k,l)=Ikl^{\mu}$ [and that of population-catalyzed asset birth is $J(k,l)=Jkl^{\nu}$], at which an aggregate of size k gains a monomer birth when it meets a catalyst aggregate of size l. The kinetic behaviors of the population and asset aggregates are solved based on the rate equations. The evolution of the aggregate size distributions of population and assets is found to fall into one of three categories for different parameters μ and ν: (i) population (asset) aggregates evolve according to the conventional scaling form in the case of $\mu\leqslant 0$ ($\nu\leqslant 0$), (ii) population (asset) aggregates evolve according to a modified scaling form in the case of $\nu=0$ and $\mu>0$ ($\mu=0$ and $\nu>0$), and (iii) both population and asset aggregates undergo gelation transitions at a finite time in the case of $\mu=\nu>0$.

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I. INTRODUCTION

Aggregation growth phenomena are popular and important in natural science for abundant kinetic evolutionary behaviors arising from a variety of complex mechanisms [1–7]. A pure aggregation process is the irreversible coalescence of clusters to form clusters of infinitely increasing mass [1,4,8–13]. This originally studied mechanism arises in diverse branches of physics, such as gelation, island formation in epitaxial surface growth, stellar evolution, and so on. Research was then steadily extended to aggregation phenomena with more complex mechanisms, such as fragmentation, annihilation, and their various combinations, and so on [14–19].

Recently, much attention has been devoted to generalized aggregation phenomena in sociology and economy to investigate the kinetic behaviors of aggregation growth driven by migration or exchange. Ispolatov, Krapivsky, and Redner introduced several asset exchange models for the evolution of wealth distribution in an economic interaction population [20]. Leyvraz and Redner proposed a migration-driven irreversible aggregate growth model for the evolution of city populations [21]. In these models, irreversible growth of aggregates takes place through biased migration or unbiased exchange mechanisms. In the biased migration model, there exists preferential migration of a monomer (equivalently, one unit of assets in asset exchange models or one person in migration-driven city population models) from a smaller aggregate to a larger aggregate. The mechanism can be described by an irreversible reaction scheme $A_k+A_l \xrightarrow{K(k,l)} A_{k-1}+A_{l+1}$ ($k\leqslant l$), where A_k denotes an aggregate characterized only by its size k (an aggregate of k units of assets in asset exchange models or k persons in migration-driven city population models) and the migration rate kernel $K(k,l)$ represents the rate of a monomer migrating from an aggregate of size k to another aggregate of size l, which generally depends on the sizes of the two aggregates, while in an unbiased exchange model, an aggregate is equally likely to gain or to lose a monomer. These processes exhibit much more abundant kinetic behaviors than those in single aggregation, annihilation, and fragmentation processes, or their various combinations.

In fact, the migration or exchange-driven aggregations exist in many branches of physics and social sciences, and their mechanisms are very complex. In our previous works, we investigated the kinetics of the general unbiased migration-driven aggregation (exchange-driven growth) system [22]. The exchange scheme is $A_k+A_l \xrightarrow{K(k,l)} A_{k-1}+A_{l+1}$ or $A_k+A_l \xrightarrow{K'(k,l)} A_{k+1}+A_{l-1}$, where $K(k,l)$ is the rate kernel of one monomer migrating from the aggregate A_k to another aggregate A_l, and $K'(k,l)$ is that of the aggregate A_k obtaining one monomer from the aggregate A_l through migration. And we further generalized the research to exchange-driven aggregation with birth and death to mimic the evolution of city population and individual wealth [23,24]. The self-birth and death schemes are $A_k \xrightarrow{J_1 k} A_{k+1}$ and $A_k \xrightarrow{J_2 k} A_{k-1}$, respectively, where J_1 and J_2 are two proportional constants of the reaction rate kernels of birth and death. Ben-Naim and Krapivsky performed general research on exchange-driven growth with the product rate kernel $K(k,l)=K(l,k)=(kl)^{\lambda}$ and the generalized homogeneous rate kernel $K(k,l)=K(l,k)=(k^{\nu}l^{\mu}+k^{\mu}l^{\nu})$ [25]. The kinetic behavior was found to fall into three categories: growth, gelation, and instant gelation.

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Generally, the exchange-driven growth of population and that of assets react with each other. A city with more assets per person will attract more migrants and its population will grow much faster. On the other hand, a city with larger population will provide more opportunities for people to develop their careers, and thus attract more migrants and more investment. In this paper, we focus on the interaction between the aggregates of population and assets in their exchange-driven aggregation processes. For simplicity, we propose a two-species (population and assets) exchange-driven aggregation growth system with their mutual catalysis-driven birth to study the interaction of the two species.

The rest of this paper is organized as follows. In Sec. II, we introduce the model of the exchange-driven aggregation growth of population and assets with their mutually catalyzed birth, and describe the outline of the generalized Smoluchowski rate equation approach to study the kinetic evolution behaviors of the aggregate size distributions of population and assets. In Sec. III, we study the kinetics of the system in various cases with different dependence of catalyzed birth rates of population and assets on the catalysts' sizes. Finally, a brief summary is given in Sec. IV.

II. MODEL OF THE EXCHANGE-DRIVEN GROWTH OF POPULATION AND ASSETS WITH THEIR MUTUALLY CATALYZED BIRTH

In this model, the exchange-driven aggregation of population is described as $A_k+A_l \xrightarrow{K(k,l)} A_{k-1}+A_{l+1}$ with exchange rate kernel $K(k,l)$, which is one person migrating from an aggregate A_k to another aggregate A_l, and thus gives rise to a gain in the number of aggregates A_{k-1} and a loss in the number of aggregates A_k at time t), which is denoted by $a_k(t)$. The exchange-driven aggregation of assets is described as $B_k+B_l \xrightarrow{L(k,l)} B_{k-1}+B_{l+1}$ with the rate kernel $L(k,l)$, which represents one unit of assets migrating from an aggregate B_k to another aggregate B_l, and gives rise to a gain in $b_{k-1}(t)$ and a loss in $b_k(t)$ at time t). In particular, the aggregate A_1 (or B_1) becomes an empty aggregate and disappears when its one unit of population (or assets) migrates out. This gives rise to a reduction in the total number of aggregates. The reaction of population birth driven by asset catalysis is $A_k+B_l \xrightarrow{I(k,l)} A_{k+1}+B_l$, with the rate kernel $I(k,l)$; and population-catalyzed asset birth is given by $A_l+B_k \xrightarrow{J(k,l)} A_l+B_{k+1}$, with the rate kernel $J(k,l)$.

In the physical sense, for asset-catalyzed population birth, the creation of more than one unit (for example, two, three or even more people) at each reaction step should be considered for a large population aggregate catalyzed by asset aggregate with a large amount of assets, and similarly for population-catalyzed asset birth. This consideration will make the system evolve in a more complicated way and the evolution behaviors become very hard to study. To perform preliminary research on such a complex mutually catalyzed birth of two species (population and assets) in exchange-driven aggregation and find the basic evolution characteristics of the aggregate size distributions, we focus here only on the one-unit catalyzed birth process; multiunit birth processes are considered as a sequence of one-unit birth processes and thus are ignored. Meanwhile, we choose the catalyzed birth rate kernels depending on the size of the aggregate that will have a monomer birth and the size of the catalyst aggregate to reflect this physical sense to some extent.

In this paper, we assume that the system has spatial homogeneity, so that fluctuations in the densities of the reactants are ignored and the aggregates are considered to be homogeneously distributed in space throughout the whole processes. Thus, the theoretical approach to investigate the kinetics of the aggregation process can be based on the rate equations in the mean-field frame, which assumes that the reaction proceeds at a rate proportional to the reactant concentrations. We generalize the rate equation of the exchange- or migration-driven aggregation process [20,21] and write the corresponding rate equations for our system as follows:

$$\frac{da_1}{dt} = \sum_{l=1}^{\infty} K(2,l)a_2a_l - \sum_{l=1}^{\infty} [K(1,l)+K(l,1)]a_1a_l - \sum_{l=1}^{\infty} I(1,l)a_1b_l, \tag{1}$$

$$\frac{da_k}{dt} = \sum_{l=1}^{\infty} K(k+1,l)a_{k+1}a_l + \sum_{l=1}^{\infty} K(l,k-1)a_{k-1}a_l - \sum_{l=1}^{\infty} [K(k,l)+K(l,k)]a_ka_l + \sum_{l=1}^{\infty} I(k-1,l)a_{k-1}b_l - \sum_{l=1}^{\infty} I(k,l)a_kb_l \quad (k=2,3,\ldots), \tag{2}$$

$$\frac{db_1}{dt} = \sum_{l=1}^{\infty} L(2,l)b_2b_l - \sum_{l=1}^{\infty} [L(1,l)+L(l,1)]b_1b_l - \sum_{l=1}^{\infty} J(1,l)b_1a_l, \tag{3}$$

$$\frac{db_k}{dt} = \sum_{l=1}^{\infty} L(k+1,l)b_{k+1}b_l + \sum_{l=1}^{\infty} L(l,k-1)b_{k-1}b_l - \sum_{l=1}^{\infty} [L(k,l)+L(l,k)]b_kb_l + \sum_{l=1}^{\infty} J(k-1,l)b_{k-1}a_l - \sum_{l=1}^{\infty} J(k,l)b_ka_l \quad (k=2,3,\ldots). \tag{4}$$

In Eq. (2) [similarly in Eq. (4)], the first two terms account for the gain in $a_k(t)$ due to the migrations $A_{k+1}+A_l \xrightarrow{K(k+1,l)} A_k+A_{l+1}$ and $A_{k-1}+A_l \xrightarrow{K(l,k-1)} A_k+A_{l-1}$ $(l=1,2,\ldots)$, while their equiprobable reaction channels $A_{k+1}+A_l \xrightarrow{K(l,k+1)} A_{k+2}+A_{l-1}$ and

$A_{k-1}+A_l \xrightarrow{K(k-1,l)} A_{k-2}+A_{l+1}$ give rise to gains in $a_{k+2}(t)$ and $a_{k-2}(t)$, which are accounted for in the rate equations da_{k+2}/dt and da_{k-2}/dt, respectively. The third term in Eqs. (2) and (4) accounts for the loss in $a_k(t)$ due to the migration $A_k+A_l \underset{K(l,k)}{\overset{K(k,l)}{\longrightarrow}} A_{k-1}+A_{l+1}$ and its equiprobable process $A_k+A_l \longrightarrow A_{k+1}+A_{l-1}$. The fourth and fifth terms account, respectively, for the gain and loss in $a_k(t)$ due to asset-catalyzed population birth.

For $a_1(t)$, the rate equation (1) has not the second and fourth terms of the equation for $a_k(t)(k=2,3,\ldots)$ because there is no empty aggregate A_0, and similarly for $b_1(t)$. But they may be written in the same forms as Eqs. (2) and (4) if we impose the boundary conditions $a_0(t)=0$ and $b_0(t)=0$.

In general, the rate kernels of exchange and catalyzed birth are dependent on the reactant aggregates' sizes. Here for the convenience of solving the rate equations, we focus on typical symmetrical exchange kernels $K(k,l)=Kkl$ and $L(k,l)=Lkl$, which are proportional to the sizes of aggregates migrating out and accepting one monomer (K and L are two proportionality constants). The catalysis-driven birth kernels of population and assets are assumed to be $I(k,l)=Ikl^{\mu}$ and $J(k,l)=Jkl^{\nu}$, respectively, where I and J are also proportionality constants, and μ and ν are parameters reflecting the dependence of the catalyzed birth rates on the catalyst aggregate size. This asset-catalyzed population birth kernel takes account of the physical sense that the next catalyzed birth in the population will take place faster for larger population and asset aggregates (when $\mu>0$). A similar effect occurs for population-catalyzed asset birth.

The rate equations for our system are then reduced to

$$\frac{da_1}{dt}=2KM_1^A(a_2-a_1)-IM_\mu^B a_1, \tag{5}$$

$$\frac{da_k}{dt}=KM_1^A[(k+1)a_{k+1}+(k-1)a_{k-1}-2ka_k] + IM_\mu^B[(k-1)a_{k-1}-ka_k] \quad (k=2,3,\ldots), \tag{6}$$

$$\frac{db_1}{dt}=2LM_1^B(b_2-b_1)-JM_\nu^A b_1, \tag{7}$$

$$\frac{db_k}{dt}=LM_1^B[(k+1)b_{k+1}+(k-1)b_{k-1}-2kb_k] + JM_\nu^A[(k-1)b_{k-1}-kb_k] \quad (k=2,3,\ldots), \tag{8}$$

where $M_\nu^A(t)=\Sigma_{k=1}^{\infty}k^{\nu}a_k(t)$ and $M_\mu^B(t)=\Sigma_{k=1}^{\infty}k^{\mu}b_k(t)$ are the νth moment of the distribution $a_k(t)$ and μth moment of $b_k(t)$, respectively. The first two moments $M_0^A(t)=\Sigma_{k=1}^{\infty}a_k(t)$ and $M_1^A(t)=\Sigma_{k=1}^{\infty}ka_k(t)$ are the total number and the total mass of the population aggregates.

Several methods have been used to solve the rate equations in different aggregation processes. For the simple coagulation process $A_i+A_j \xrightarrow{K(i,j)} A_{i+j}$, explicit solutions were obtained for $K(i,j)=1$, $i+j$, and ij by introducing some suitable generating functions [1,4,8,9]. In aggregation-annihilation processes, the rate equations were solved with the help of a special *Ansatz* $a_k(t)=A(t)[a(t)]^{k-1}$, and by the generating function method as well [15]. In exchange-driven aggregation processes, the rate equations were solved through making some scaling *Ansatz* directly [20,21,25] or with the help of the special *Ansatz* $a_k(t)=A(t)[a(t)]^{k-1}$ [22–24].

Here, for the rate equation (6), we find that if we make the assumption of a recursion relation $a_k(t)=a(t)a_{k-1}(t)$ ($k=2,3,\ldots$), the rate equation for $a_k(t)$ can be reduced to a rate equation for $a_{k-1}(t)$ with the function $a(t)$ governed by the equation

$$\frac{da}{dt}=KM_1^A(1-a)^2+IM_\mu^B(1-a), \tag{9}$$

which is valid for all rate equations of $a_k(t)$ ($k=2,3,\ldots$). So in this paper, we rewrite the above recursion relation assumption for $a_k(t)$ in an *Ansatz* form, and correspondingly for $b_k(t)$, as

$$a_k(t)=a_1(t)[a(t)]^{k-1}, \quad b_k(t)=b_1(t)[b(t)]^{k-1}. \tag{10}$$

These equations were used in solving the rate equations of aggregation-annihilation processes by Krapivsky [15] and in our previous work [22–24] [here we use $a_1(t)$ instead of $A(t)$ to make its meaning more explicit]. The rate equations are solved under monodisperse initial conditions, which assume that there exist only monomer aggregates of population and assets at $t=0$ (more precisely, at $t=0^+$). They can be expressed as $a_k(0)=A_0\delta_{k1}$ and $b_k(0)=B_0\delta_{k1}$, where A_0 and B_0 are the initial aggregate concentrations of population and assets, respectively.

Substituting the *Ansatz* (10) into the rate equation (5), we derive the differential equation for $a_1(t)$ as

$$\frac{da_1}{dt}=-2KM_1^A a_1(1-a)-IM_\mu^B a_1. \tag{11}$$

When we substitute the *Ansatz* (10) into the rate equation (6), it is transformed into

$$a^{k-1}\frac{da_1}{dt}+(k-1)a_1a^{k-2}\frac{da}{dt} = a^{k-1}[-2KM_1^A a_1(1-a)-IM_\mu^B a_1] + (k-1)a_1a^{k-2}[KM_1^A(1-a)^2+IM_\mu^B(1-a)] \quad (k=2,3,\ldots). \tag{12}$$

Because this equation is valid for all $k=2,3,\ldots$, it is separated into two differential equations for $a(t)$ and $a_1(t)$, which are the same as Eqs. (9) and (11), respectively. Here with the help of the recursion relation assumption or *Ansatz* (10), the infinite set of rate equations for $a_1(t)$ and $a_k(t)$ ($k=2,3,\ldots$) are reduced to the same two ordinary differential equations.

The infinite set of rate equations for $b_1(t)$ and $b_k(t)$ ($k=2,3,\ldots$) are also reduced to two ordinary differential equations under the *Ansatz* (10) in the same way as

$$\frac{db}{dt} = LM_1^B(1-b)^2 + JM_\nu^A(1-b), \tag{13}$$

$$\frac{db_1}{dt} = -2LM_1^B b_1(1-b) - JM_\nu^A b_1. \tag{14}$$

The monodisperse initial conditions are transformed correspondingly into

$$a=0, \quad a_1=A_0, \quad b=0, \quad b_1=B_0 \quad \text{at } t=0. \tag{15}$$

From the viewpoint of mathematics, the use of the *Ansatz* (10) is to give restrictions $a_k(t)=a(t)a_{k-1}(t)$ $(k=2,3,\ldots)$ and $b_k(t)=b(t)b_{k-1}(t)$ $(k=2,3,\ldots)$ on the solutions of the original rate equations (5)–(8). So the solutions of $a_k(t)$ and $b_k(t)$ from the reduced differential equations (9), (11), (13), and (14) are special solutions of the original rate equations (5)–(8). This method is popular in solving differential equations. When the added restrictions are consistent with the original differential equations, special solutions can be obtained, otherwise no special solutions can be obtained. In the following, we can solve the reduced differential equations (9), (11), (13), and (14) to obtain the special solutions of the original rate equations (5)–(8). This indicates that in this model the *Ansatz* (10) or the restrictions $a_k(t)=a(t)a_{k-1}(t)$ $(k=2,3,\ldots)$ and $b_k(t)=b(t)b_{k-1}(t)$ $(k=2,3,\ldots)$ are consistent with the original rate equations, and thus the *Ansatz* (10) are applicable.

In the *Ansatz* (10), the first two moments of the aggregate size distributions $a_k(t)$ and $b_k(t)$ can be expressed as

$$M_0^A(t) = \sum_{k=1}^{\infty} a_k(t) = a_1 \sum_{k=1}^{\infty} a^{k-1} = \frac{a_1}{1-a}, \tag{16}$$

$$M_1^A(t) = \sum_{k=1}^{\infty} k a_k(t) = a_1 \sum_{k=1}^{\infty} k a^{k-1} = \frac{a_1}{(1-a)^2}, \tag{17}$$

$$M_0^B(t) = \sum_{k=1}^{\infty} b_k(t) = b_1 \sum_{k=1}^{\infty} b^{k-1} = \frac{b_1}{1-b}, \tag{18}$$

$$M_1^B(t) = \sum_{k=1}^{\infty} k b_k(t) = b_1 \sum_{k=1}^{\infty} k b^{k-1} = \frac{b_1}{(1-b)^2}. \tag{19}$$

Using these moment expressions, Eqs. (9), (11), (13), and (14) can be rewritten as

$$\frac{1}{1-a}\frac{da}{dt} = KM_0^A + IM_\mu^B, \tag{20}$$

$$\frac{1}{a_1}\frac{da_1}{dt} = -2KM_0^A - IM_\mu^B, \tag{21}$$

$$\frac{1}{1-b}\frac{db}{dt} = LM_0^B + JM_\nu^A, \tag{22}$$

$$\frac{1}{b_1}\frac{db_1}{dt} = -2LM_0^B - JM_\nu^A. \tag{23}$$

From these equations, the equations for the total aggregate numbers of population and assets can be derived as follows:

$$\frac{d\ln M_0^A}{dt} = \frac{1}{1-a}\frac{da}{dt} + \frac{1}{a_1}\frac{da_1}{dt} = -KM_0^A, \tag{24}$$

$$\frac{d\ln M_0^B}{dt} = \frac{1}{1-b}\frac{db}{dt} + \frac{1}{b_1}\frac{db_1}{dt} = -LM_0^B. \tag{25}$$

These equations are solved under the initial conditions $M_\nu^A(0)=\Sigma_{k=1}^{\infty} k^\nu a_k(0)=\Sigma_{k=1}^{\infty} k^\nu A_0 \delta_{k1}=A_0$ and $M_\mu^B(0)=B_0$ to yield

$$M_0^A = A_0(1+KA_0t)^{-1}, \tag{26}$$

$$M_0^B = B_0(1+LB_0t)^{-1}. \tag{27}$$

The differential equations for M_1^A and M_1^B are also derived from Eqs. (20)–(23) by the use of the M_1^A and M_1^B expressions,

$$\frac{d\ln M_1^A}{dt} = \frac{2}{1-a}\frac{da}{dt} + \frac{1}{a_1}\frac{da_1}{dt} = IM_\mu^B, \tag{28}$$

$$\frac{d\ln M_1^B}{dt} = JM_\nu^A. \tag{29}$$

The solutions of those equations are

$$M_1^A = A_0 \exp\left(I\int_0^t M_\mu^B dt\right), \tag{30}$$

$$M_1^B = B_0 \exp\left(J\int_0^t M_\nu^A dt\right). \tag{31}$$

In the next section we study the kinetics of the system in various cases with different catalyzed birth rate kernel parameters μ and ν, which reflect the dependence of the catalyzed birth rates on the catalyst aggregate sizes.

III. KINETICS OF THE POPULATION AND ASSET MUTUALLY CATALYZED BIRTH MODEL

A. The case with $\mu=0$ and $\nu=0$

First, we investigate the case with catalyzed birth rate kernel parameters $\mu=0$ and $\nu=0$, a case where two catalyzed birth rates $I(k,l)=Ik$ and $J(k,l)=Jk$ are both independent of the catalyst aggregate sizes.

Substituting Eqs. (26) and (27) into Eq. (20), we derive the solution for $a(t)$,

$$a(t) = 1-(1+KA_0t)^{-1}(1+LB_0t)^{-I/L}. \tag{32}$$

Using the expression $a_1=M_0^A(1-a)$, the population aggregate size distribution can then be solved exactly,

$$a_k(t) = A_0(1 + KA_0 t)^{-2}(1 + LB_0 t)^{-I/L} \times [1 - (1 + KA_0 t)^{-1}(1 + LB_0 t)^{-I/L}]^{k-1}. \tag{33}$$

It approaches the conventional scaling form in the scaling region of $k \gg 1$ and $t \gg 1$ [15]:

$$a_k(t) \simeq K^{-2}A_0^{-1}(LB_0)^{-I/L}t^{-2-I/L}\Phi(x), \quad x = k/S_A(t), \tag{34}$$

with the scaling function $\Phi(x)=\exp(-x)$. The characteristic size of the population aggregates is

$$S_A(t) = KA_0(LB_0)^{I/L}t^{1+I/L}. \tag{35}$$

It grows with time in power law form.

Here a very important aggregation phenomenon is found—the kinetic behavior of $a_k(t)$ is dominated by asset exchange and asset-catalyzed population birth. But it is unexpectedly not dominated by the exchange of population itself.

The total population can be obtained as

$$M_1^A = a_1(1 - a)^{-2} = A_0(1 + LB_0 t)^{I/L}. \tag{36}$$

It grows with time as a power law in the long-time limit.

The conventional scaling expression of population aggregate size distribution (34) can be modified as follows:

$$a_k(t) \simeq K^{-1}t^{-1}[S_A(t)]^{-1}\Phi(x), \quad x = k/S_A(t). \tag{37}$$

Here it is worth noticing that the aggregate size distribution can be written as $a_k(t)=M_0^A(1-a)a^{k-1}$ under the *Ansatz* (10). So if the solution of $a(t)$ can be written as $a(t)=1-[S_A(t)]^{-1}$ and $S_A(t)$ grows with time monotonically, the aggregate size distribution can be described in the following modified scaling form:

$$a_k(t) \simeq M_0^A(t)[S_A(t)]^{-1} \exp(-x), \quad x = k/S_A(t), \tag{38}$$

where $S_A(t)$ is the characteristic size of the aggregates.

The asset aggregate distribution can be obtained in the same way,

$$b_k(t) = B_0(1 + LB_0 t)^{-2}(1 + KA_0 t)^{-J/K} \times [1 - (1 + LB_0 t)^{-1}(1 + KA_0 t)^{-J/K}]^{k-1}, \tag{39}$$

and it obeys the same conventional scaling form,

$$b_k(t) \simeq L^{-2}B_0^{-1}(KA_0)^{-J/K}t^{-2-J/K}\Phi(x), \quad x = k/S_B(t), \tag{40}$$

with the scaling function $\Phi(x)=\exp(-x)$ and the characteristic size of the asset aggregates $S_B(t)=LB_0(KA_0)^{J/K}t^{1+J/K}$. The kinetic behavior of $b_k(t)$ is found to be dominated by population exchange and population-catalyzed asset birth. It is also unexpectedly not dominated by the exchange of an asset itself.

If $S_B(t)$ grows with time monotonically, the aggregate size distribution $b_k(t)$ also can be described in the following modified scaling form:

$$b_k(t) \simeq L^{-1}t^{-1}[S_B(t)]^{-1}\Phi(x) \simeq M_0^B(t)[S_B(t)]^{-1} \exp(-x), \quad x = k/S_B(t). \tag{41}$$

Because $M_0^A(t)$ and $M_0^B(t)$ remain unchanged for all cases with different values of the parameters μ and ν, the aggregate size distributions $a_k(t)$ and $b_k(t)$ will keep the same scaling forms as Eq. (37) and (41), respectively. The different values of parameters μ and ν bring changes only in the characteristic sizes of the aggregates $S_A(t)$ and $S_B(t)$.

B. The case with $\mu=0$ and general ν

We then study the $\mu=0$, $\nu=1$ case (the $\mu=1$, $\nu=0$ case has symmetric solutions). In this case, the asset-catalyzed population birth rate kernel $I(k,l)=Ik$ is still independent of the catalyst (asset) aggregate's size as in the above case, while the population-catalyzed asset birth rate kernel $J(k,l)=Jkl$ is proportional to the catalyst (population) aggregate's size. Because the rate equations (5) and (6) for $a_1(t)$ and $a_k(t)$, and the differential equations (20) and (21) for $a(t)$ and $a_1(t)$ are the same for all cases with the same μ and different ν, the population aggregate size distributions $a_k(t)$ for all cases with the same μ and different ν are also expressed as Eqs. (33) and (34).

For the asset aggregates, we derive the equation for $b(t)$ by substituting Eqs. (27) and (36) into Eq. (22),

$$\frac{1}{1-b}\frac{db}{dt} = LB_0(1 + LB_0 t)^{-1} + JA_0(1 + LB_0 t)^{I/L}. \tag{42}$$

The solution is obtained under the initial condition (15) as

$$b(t) = 1 - \exp\left(\frac{JA_0}{(I+L)B_0}\right)(1 + LB_0 t)^{-1} \times \exp\left(-\frac{JA_0}{(I+L)B_0}(1 + LB_0 t)^{1+I/L}\right). \tag{43}$$

Using the expression $b_1=M_0^B(1-b)$, we obtain the exact solution of the asset aggregate size distribution:

$$b_k(t) = \frac{B_0 \exp\left[\frac{JA_0}{(I+L)B_0}\right]}{(1 + LB_0 t)^2 \exp\left[\frac{JA_0}{(I+L)B_0}(1 + LB_0 t)^{1+I/L}\right]} \times \left(1 - \frac{\exp\left[\frac{JA_0}{(I+L)B_0}\right]}{(1 + LB_0 t)\exp\left[\frac{JA_0}{(I+L)B_0}(1 + LB_0 t)^{1+I/L}\right]}\right)^{k-1}. \tag{44}$$

Its asymptotic behavior can be written in a scaling form,

$$b_k(t) \simeq L^{-2}B_0^{-1} \exp\left(\frac{JA_0}{(I+L)B_0}\right)t^{-2} \times \exp\left(-\frac{JA_0}{(I+L)B_0}(LB_0 t)^{1+I/L}\right)\Phi(x), \quad x = k/S_B(t), \tag{45}$$

with the scaling function $\Phi(x)=\exp(-x)$, where the typical size of the asset aggregates is

$$S_B(t) = LB_0 \exp\left(-\frac{JA_0}{(I+L)B_0}\right) t \exp\left(\frac{JA_0}{(I+L)B_0}(LB_0 t)^{1+I/L}\right). \tag{46}$$

This asset aggregate size distribution may also be written in a modified scaling form as Eq. (41).

Here the kinetic behavior of $b_k(t)$ is dominated by the asset exchange and the catalyzed birth of population and assets, not by the exchange of population, in contrast to the above $\mu=\nu=0$ case.

The total mass of the assets can be obtained as

$$M_1^B = \frac{M_0^B}{1-b} = B_0 \exp\left(-\frac{JA_0}{(I+L)B_0}\right) \times \exp\left(\frac{JA_0}{(I+L)B_0}(1+LB_0 t)^{1+I/L}\right). \tag{47}$$

It grows with time exponentially in the long-time limit.

The $\mu=0$ and general ν case can be studied in the long-time region. From the population aggregate size distribution (33), we can calculate its moments in the long-time limit as follows:

$$M_\nu^A = \sum_{k=1}^{\infty} k^\nu a_k \simeq \begin{cases} c_1 t^{-1+(1+I/L)\nu} & \text{for } \nu > -1, \\ c_2 t^{-2-I/L} \ln t & \text{for } \nu = -1, \\ c_3 t^{-2-I/L} & \text{for } \nu < -1, \end{cases} \tag{48}$$

where

$$c_1 = \Gamma(1+\nu) K^{-1} (KA_0)^\nu (LB_0)^{I\nu/L},$$

$$c_2 = (1+I/L) A_0 (KA_0)^{-2} (LB_0)^{-I/L},$$

$$c_3 = \zeta(-\nu) A_0 (KA_0)^{-2} (LB_0)^{-I/L},$$

and $\zeta(n)=\sum_{l=1}^{\infty} l^{-n}$ $(n>1)$ is the Riemann zeta function.

Inserting M_ν^A into the governing equation (22), we find the solution for $b(t)$ as follows:

$$b \simeq \begin{cases} 1-(LB_0 t)^{-1} \exp(-c_4 t^{(1+I/L)\nu}) & \text{for } \nu > 0, \\ 1-(LB_0 t)^{-1} & \text{for } \nu < 0, \end{cases} \tag{49}$$

where $c_4 = JLc_1/(I+L)\nu$. The asset aggregate size distribution is then obtained. It evolves according to the conventional scaling form in the $\nu<0$ case, and it evolves according to the modified scaling form in the $\nu>0$ case as Eq. (41) with the characteristic aggregate size

$$S_B(t) = \begin{cases} LB_0 t \exp(c_4 t^{(1+I/L)\nu}) & \text{for } \nu > 0, \\ LB_0 t & \text{for } \nu < 0. \end{cases} \tag{50}$$

C. The case with general $\mu=\nu$

We now study the $\mu=\nu=1$ case, where the catalyzed birth rate kernels $I(k,l)=Ikl$ and $J(k,l)=Jkl$ are both proportional to the catalyst aggregate sizes. Equations (28) and (29) become

$$\frac{d \ln M_1^A}{dt} = IM_1^B, \tag{51}$$

$$\frac{d \ln M_1^B}{dt} = JM_1^A. \tag{52}$$

The relation between M_1^A and M_1^B can be derived directly under the initial conditions $M_1^A(0)=A_0$ and $M_1^B(0)=B_0$, i.e.,

$$J(M_1^A - A_0) = I(M_1^B - B_0). \tag{53}$$

With this relation, Eq. (51) can be written as

$$\frac{d \ln M_1^A}{dt} = J(M_1^A - A_0) + IB_0. \tag{54}$$

For the $JA_0 \neq IB_0$ case, it can be solved exactly to yield

$$M_1^A = A_0 \frac{JA_0 - IB_0}{JA_0 - IB_0 e^{(JA_0 - IB_0)t}}. \tag{55}$$

In contrast to all the above cases, here the solution is valid only in the time region $t<t_c$, where

$$t_c = \frac{\ln(JA_0/IB_0)}{JA_0 - IB_0}. \tag{56}$$

The total mass of asset aggregates M_1^B can be solved exactly in the same way from Eqs. (52) and (55). The solution is

$$M_1^B = B_0 \frac{IB_0 - JA_0}{IB_0 - JA_0 e^{(IB_0 - JA_0)t}}, \tag{57}$$

which is also valid only in the time region of $t<t_c$.

The evolution equations of population and asset aggregates and their solutions reveal another important behavior of the aggregation. M_1^A and M_1^B grow with time as in the above cases, but they grow much faster because the catalyzed birth rate kernels of population and assets are both proportional to the size of the aggregate (population aggregate or asset aggregate) itself, as well as the catalyst aggregate sizes. When t increases and reaches t_c, M_1^A and M_1^B approach infinite values. When t reaches t_c, their kinetic behaviors can be analyzed. For example, at $t=t_c-\Delta t$ $(\Delta t \to 0^+)$, M_1^A can be written as

$$M_1^A(t_c - \Delta t) = A_0 \frac{JA_0 - IB_0}{JA_0(1-e^{-(JA_0 - IB_0)\Delta t})} \simeq \frac{A_0}{JA_0 \Delta t}. \tag{58}$$

So at a finite time $t=t_c$, gelation transitions of population and asset aggregates take place.

In the time region $t<t_c$, we study the evolution behavior of population and asset aggregates approaching the gelation point. Substituting Eqs. (26), (27), (55), and (57) into Eqs. (20) and (22), we derive the differential equations for $a(t)$ and $b(t)$ as follows:

$$\frac{1}{1-a}\frac{da}{dt} = KA_0(1+KA_0 t)^{-1} + IB_0 \frac{IB_0 - JA_0}{IB_0 - JA_0 e^{(IB_0 - JA_0)t}}, \tag{59}$$

$$\frac{1}{1-b}\frac{db}{dt} = LB_0(1+LB_0t)^{-1} + JA_0\frac{JA_0-IB_0}{JA_0-IB_0e^{(JA_0-IB_0)t}}. \tag{60}$$

These equations can be solved exactly under the initial conditions (15). The results are

$$a = 1 - \frac{JA_0 - IB_0e^{(JA_0-IB_0)t}}{(JA_0-IB_0)(1+KA_0t)}, \tag{61}$$

$$b = 1 - \frac{IB_0 - JA_0e^{(IB_0-JA_0)t}}{(IB_0-JA_0)(1+LB_0t)}. \tag{62}$$

Using the expressions $a_1=M_0^A(1-a)$ and $b_1=M_0^B(1-b)$, we obtain the kinetic behaviors of population and asset aggregate distributions before the gelation point,

$$a_k(t) = \frac{A_0}{JA_0-IB_0}\frac{JA_0-IB_0e^{(JA_0-IB_0)t}}{(1+KA_0t)^2} \times\left(1-\frac{JA_0-IB_0e^{(JA_0-IB_0)t}}{(JA_0-IB_0)(1+KA_0t)}\right)^{k-1}, \tag{63}$$

$$b_k(t) = \frac{B_0}{IB_0-JA_0}\frac{IB_0-JA_0e^{(IB_0-JA_0)t}}{(1+LB_0t)^2} \times\left(1-\frac{IB_0-JA_0e^{(IB_0-JA_0)t}}{(IB_0-JA_0)(1+LB_0t)}\right)^{k-1}. \tag{64}$$

For the $JA_0=IB_0$ case, the relation between M_1^A and M_1^B of Eq. (53) becomes

$$JM_1^A = IM_1^B. \tag{65}$$

The solutions of M_1^A and M_1^B from Eqs. (51) and (52) are

$$M_1^A = A_0(1-JA_0t)^{-1}, \tag{66}$$

$$M_1^B = B_0(1-IB_0t)^{-1}. \tag{67}$$

These are also valid only for time $t<t_c=(JA_0)^{-1}=(IB_0)^{-1}$. When t increases to t_c, the population and asset aggregates approach infinite gelation clusters.

In the time region $t<t_c$, the evolution behavior of population and asset aggregates before the gelation point can be obtained in the same way,

$$a_k(t) = A_0\frac{1-JA_0t}{(1+KA_0t)^2}\left(1-\frac{1-JA_0t}{1+KA_0t}\right)^{k-1}, \tag{68}$$

$$b_k(t) = B_0\frac{1-IB_0t}{(1+LB_0t)^2}\left(1-\frac{1-IB_0t}{1+LB_0t}\right)^{k-1}. \tag{69}$$

The general $\mu=\nu$ case can be studied in the long-time region. On the *Ansatz* (10), we derive the expressions of population and asset aggregate size distributions by their first two moments. For the population aggregates, $a(t)$ and $a_1(t)$ are directly expressed as follows:

$$a = 1 - \frac{M_0^A}{M_1^A}, \quad a_1 = \frac{(M_0^A)^2}{M_1^A}. \tag{70}$$

The population aggregate size distribution is written as

$$a_k = \frac{(M_0^A)^2}{M_1^A}\left(1-\frac{M_0^A}{M_1^A}\right)^{k-1}, \tag{71}$$

and that of asset aggregate can be written in the corresponding form.

Considering the effects of the exchange-driven aggregation and the catalyzed birth, we can derive $M_0^A/M_1^A \ll 1$ in the long-time limit. The νth moment of the population aggregate size distribution can be derived in the long-time limit as follows:

$$M_\nu^A \simeq \begin{cases} \Gamma(1+\nu)\dfrac{(M_1^A)^\nu}{(M_0^A)^{\nu-1}} & \text{for } \nu>-1, \\ \dfrac{(M_0^A)^2}{M_1^A}\left(1-\dfrac{M_0^A}{M_1^A}\right)^{-1}\ln\dfrac{M_1^A}{M_0^A} & \text{for } \nu=-1, \\ \dfrac{(M_0^A)^2}{M_1^A}\zeta(-\nu) & \text{for } \nu<-1. \end{cases} \tag{72}$$

That of the asset aggregate has the corresponding form.

Now we turn to derive the relation between M_1^A and M_1^B. From Eqs. (28) and (29), we have

$$JM_\nu^A(M_1^A)^{-1}dM_1^A = IM_\mu^B(M_1^B)^{-1}dM_1^B. \tag{73}$$

For the $\mu=\nu>-1$ case, Eq. (73) is rewritten as follows using the M_μ^A and M_μ^B expressions (72):

$$J\frac{(M_1^A)^{\mu-1}}{(M_0^A)^{\mu-1}}dM_1^A \simeq I\frac{(M_1^B)^{\mu-1}}{(M_0^B)^{\mu-1}}dM_1^B. \tag{74}$$

Using Eqs. (26) and (27), it can be further written as

$$JK^{\mu-1}(M_1^A)^{\mu-1}dM_1^A \simeq IL^{\mu-1}(M_1^B)^{\mu-1}dM_1^B. \tag{75}$$

When $\mu\neq 0$, it can be solved to yield

$$JK^{\mu-1}(M_1^A)^\mu \simeq IL^{\mu-1}(M_1^B)^\mu. \tag{76}$$

Hence we solve for M_1^B and further obtain M_μ^B as

$$M_\mu^B \simeq \Gamma(1+\mu)\frac{J}{I}\left(\frac{K}{L}\right)^{\mu-1}\frac{(M_1^A)^\mu}{(M_0^B)^{\mu-1}}. \tag{77}$$

Inserting it into Eq. (28) and using the M_0^B solution (27), we derive the equation for M_1^A as follows:

$$(M_1^A)^{-\mu-1}dM_1^A \simeq \Gamma(1+\mu)JK^{\mu-1}t^{\mu-1}dt. \tag{78}$$

For $\mu>0$, the total mass of the population M_1^A is then finally obtained in the long-time limit,

$$\begin{aligned} M_1^A &\simeq [\Gamma(1+\mu)JK^{\mu-1}]^{-1/\mu}[(t_c^A)^\mu - t^\mu]^{-1/\mu} \\ &\simeq [\Gamma(1+\mu)\mu JK^{\mu-1}]^{-1/\mu}(t_c^A)^{-(\mu-1)/\mu}(t_c^A-t)^{-1/\mu}, \end{aligned} \tag{79}$$

where t_c^A is a finite critical time when $M_1^A(t_c^A)\to\infty$. The total mass of the assets in the long-time limit is obtained in the same way as

$$M_1^B \simeq [\Gamma(1+\mu)\mu I L^{\mu-1}]^{-1/\mu}(t_c^B)^{-(\mu-1)/\mu}(t_c^B - t)^{-1/\mu}. \quad (80)$$

But using the solution of M_1^A, we derive M_1^B from Eq. (76) as

$$M_1^B \simeq [\Gamma(1+\mu)\mu I L^{\mu-1}]^{-1/\mu}(t_c^A)^{-(\mu-1)/\mu}(t_c^A - t)^{-1/\mu}. \quad (81)$$

So M_1^A and M_1^B reach infinite values at the same time $t_c = t_c^A = t_c^B$, which reveals that the population and asset aggregates undergo gelation transitions at the same gelation time t_c.

For $-1 < \mu < 0$, Eq. (78) gives

$$M_1^A \simeq [-\Gamma(1+\mu)JK^{\mu-1}t^{\mu} - c_6\mu]^{-1/\mu} \simeq (-c_6\mu)^{-1/\mu}, \quad (82)$$

where c_6 is an integration constant. From Eq. (76) we have

$$M_1^B \simeq \left(\frac{J}{I}\right)^{1/\mu}\left(\frac{K}{L}\right)^{(\mu-1)/\mu} M_1^A. \quad (83)$$

So M_1^A and M_1^B grow with time and reach constant values in the long-time limit. The μth moments of population and assets are then derived as follows:

$$M_\mu^A \simeq \Gamma(1+\mu)\frac{(M_1^A)^{\mu}}{(M_0^A)^{\mu-1}} \simeq \Gamma(1+\mu)K^{\mu-1}(M_1^A)^{\mu}t^{\mu-1}, \quad (84)$$

$$M_\mu^B \simeq \Gamma(1+\mu)\frac{(M_1^B)^{\mu}}{(M_0^B)^{\mu-1}} \simeq \Gamma(1+\mu)L^{\mu-1}(M_1^B)^{\mu}t^{\mu-1}. \quad (85)$$

Inserting Eq. (85) into Eq. (20), we derive the equation for $a(t)$ as

$$\frac{1}{1-a}\frac{da}{dt} = t^{-1} + \Gamma(1+\mu)IL^{\mu-1}(M_1^B)^{\mu}t^{\mu-1} \simeq t^{-1}. \quad (86)$$

The solution of this equation is

$$a(t) \simeq 1 - t^{-1}. \quad (87)$$

With the expression $a_1 = M_0^A(1-a)$, the population aggregate size distribution in the long-time limit can then be obtained, obeying the conventional scaling form,

$$a_k(t) \simeq K^{-1}t^{-2}\exp[-k/S_A(t)], \quad S_A(t) \simeq t. \quad (88)$$

The asset aggregate size distribution obeys the same conventional scaling form with the same scaling exponents,

$$b_k(t) \simeq L^{-1}t^{-2}\exp[-k/S_B(t)], \quad S_B(t) \simeq t. \quad (89)$$

For $\mu = \nu = -1$, we derive the equation for the relation between M_1^A and M_1^B in the long-time limit from Eqs. (28) and (29) by the M_1^A and M_1^B expressions (72),

$$J\left(\frac{M_0^A}{M_1^A}\right)^2 \ln M_1^A dM_1^A \simeq I\left(\frac{M_0^B}{M_1^B}\right)^2 \ln M_1^B dM_1^B. \quad (90)$$

Using the M_0^A and M_0^B expressions, we further write the equation as

$$JK^{-2}(M_1^A)^{-2}\ln M_1^A dM_1^A \simeq IL^{-2}(M_1^B)^{-2}\ln M_1^B dM_1^B. \quad (91)$$

This gives the relation between M_1^A and M_1^B as

$$JK^{-2}(M_1^A)^{-1}\ln M_1^A \simeq IL^{-2}(M_1^B)^{-1}\ln M_1^B. \quad (92)$$

From this we rewrite M_μ^B as

$$M_\mu^B \simeq \frac{(M_0^B)^2}{M_1^B}\ln M_1^B \simeq I^{-1}JK^{-2}t^{-2}(M_1^A)^{-1}\ln M_1^A. \quad (93)$$

Inserting it into Eq. (28), we derive the equation for M_1^A as

$$(\ln M_1^A)^{-1}dM_1^A \simeq JK^{-2}t^{-2}dt. \quad (94)$$

The integration of this equation gives the equation that M_1^A satisfies as

$$\ln(\ln M_1^A) + \ln M_1^A + \frac{1}{2\times 2!}(\ln M_1^A)^2 + \frac{1}{3\times 3!}(\ln M_1^A)^3 + \cdots \simeq -JK^{-2}t^{-1} + c_7 \simeq c_7, \quad (95)$$

where c_7 is an integration constant. So M_1^A grows with time and reaches a constant value in the long-time limit, and so does M_1^B. The population and asset aggregate size distributions in the long-time limit obey the same conventional scaling form as those in the $-1 < \mu < 0$ case.

For $\mu = \nu < -1$, we again derive the equation for the relation between M_1^A and M_1^B in the long-time limit from Eqs. (28) and (29) by the expressions for M_0^A, M_0^B, M_1^A, and M_1^B,

$$JK^{-2}(M_1^A)^{-2}dM_1^A \simeq IL^{-2}(M_1^B)^{-2}dM_1^B. \quad (96)$$

It gives the relation between M_1^A and M_1^B,

$$JK^{-2}(M_1^A)^{-1} \simeq IL^{-2}(M_1^B)^{-1} + c_8, \quad (97)$$

where c_8 is an integration constant. Using the solution of M_1^B to express M_μ^B and inserting it into Eq. (28), we transform the equation of M_1^A into

$$(1 - c_8 M_1^A)^{-1}dM_1^A \simeq \zeta(-\mu)JK^{-2}t^{-2}dt. \quad (98)$$

The solution is

$$M_1^A \simeq c_8^{-1}\{1 - c_9\exp[c_8\zeta(-\mu)JK^{-2}t^{-1}]\} \simeq c_8^{-1}, \quad (99)$$

where c_9 is an integration constant. So M_1^A and M_1^B again reach constant values in the long-time limit. The population and asset aggregate size distributions in the long-time limit again obey the same conventional scaling form as in the cases $-1 < \mu = \nu < 0$ and $\mu = \nu = -1$ above.

In conclusion, we obtain the evolution behaviors of our system in the general $\mu = \nu$ case. The population and asset aggregate size distributions evolve according to the conventional scaling form in the $\mu = \nu < 0$ case, and the total masses M_1^A and M_1^B grow with time and reach constant values in the long-time limit. When $\mu = \nu > 0$, both the population and asset aggregates undergo gelation transitions at a finite time, which does not appear in the migration-driven aggregation process with self-birth and -death [23].

IV. SUMMARY

In summary, we have studied the interaction between the population and assets in exchange-driven processes through

TABLE I. Summary of the kinetic behavior of the system in different cases.

Case	Scaling behaviors of both species
$\mu=\nu<0$	(i) Both the population and asset aggregates evolve according to the conventional scaling form.
$\mu=0$	(i) The population aggregates evolve according to the conventional scaling form.
	(ii) If $\nu\leq 0$, the asset aggregates evolve according to the conventional scaling form.
	(iii) If $\nu>0$, the asset aggregates evolve according to the modified scaling form.
$\nu=0$	(i) The asset aggregates evolve according to the conventional scaling form.
	(ii) If $\mu\leq 0$, the population aggregates evolve according to the conventional scaling form.
	(iii) If $\mu>0$, the population aggregates evolve according to the modified scaling form.
$\mu=\nu>0$	(i) Both the population and asset aggregates undergo gelation transitions at a finite time.

an exchange-driven aggregation growth model of population and assets with their mutually catalyzed birth. The rate kernel of population exchange-driven growth, $K(k,l)=Kkl$, and that of assets, $L(k,l)=Lkl$, are both proportional to the sizes of aggregates migrating out and accepting one monomer. The rate kernel of asset-catalyzed birth of population is $I(k,l)=Ikl^{\mu}$ and that of population-catalyzed birth of assets is $J(k,l)=Jkl^{\nu}$. Based on the mean-field theory, we investigated the evolution behaviors of the aggregate size distributions of the population and assets by solving their generalized Smoluchowski rate equations. The kinetic behaviors are found to fall into three categories, which are illustrated in Table I. (i) When the rate kernels of the asset-catalyzed birth of population and population-catalyzed birth of assets are both independent of the catalyst aggregate sizes, i.e., $I(k,l)=Ik$ and $J(k,l)=Jk$, the aggregate size distributions of the two species obey the conventional scaling law. The kinetic behavior of population aggregate size distribution $a_k(t)$ is dominated by asset exchange and asset-catalyzed population birth, and is unexpectedly not dominated by the exchange of population itself. On the other hand, the kinetic behavior of asset aggregate size distribution $b_k(t)$ is dominated by population exchange and population-catalyzed asset birth. (ii) When the asset-catalyzed population birth rate kernel $I(k,l)=Ik$ is still independent of the catalyst (asset) aggregate's size while the population-catalyzed asset birth rate kernel $J(k,l)=Jkl^{\nu}$ $(\nu\neq 0)$ is dependent on the catalyst (population) aggregate's size, the aggregate size distribution of population remains unchanged as in case (i), while the asset aggregate size distribution obeys a modified scaling law. (iii) When the rate kernels of asset-catalyzed population birth and population-catalyzed asset birth are both dependent on the sizes of the catalyst aggregates, $I(k,l)=Ikl^{\mu}$ and $J(k,l)=Jkl^{\nu}$ $(\mu=\nu\neq 0)$, both the population and asset aggregates undergo gelation transitions at a finite time in the $\mu=\nu>0$ case, while in the $\mu=\nu<0$ case, the total population and the total mass of assets approach finite values, and their aggregate size distributions obey the conventional scaling form. Meanwhile in the migration-driven aggregation process with self-birth and -death, there is no gelation transition at finite time.

On the other hand, we also make a qualitative connection between our theoretical prediction and realistic data of urban population of a country. It is well known that the total size of a country grows exponentially with time. Similarly to Eq. (50), the typical population size increases as $\exp(ct^{(1+J/K)\mu})$ for $\mu>0$. Thus, if $(1+J/K)\mu=1$, the typical population size grows as $\exp(ct)$. Since the total size and the typical size of the population have the same order of magnitude, the total population size indeed grows exponentially in the case of $(1+J/K)\mu=1$ and $\mu>0$.

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